# Mathematical Billiards 

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Pool or billiards:
Spherical balls of different colors are on the surface of a rectangular tabletop. That table has boundary cushions mounted on the four rails (low walls).
The player hits the white cue ball with the end of the cue stick.
The cue ball rolls across the table.
Cue ball might bounce against several rails or hit another ball.
Today's discussions mostly involves a cue ball (white) and one object ball (red) on a billiard table.

## First question:

Where should the player aim the cue ball so that it will bounce against a cushion and then hit the object ball?

This is an example of a kick shot (cue ball bounces against a rail before it hits the object ball).


Our pictures are views of the table from above. Rails (cushions) are indicated by dark green lines.

Dotted lines indicate the path of the cue ball.

This problem leads to a mathematical model.
Make simplifying assumptions to get an abstract math problem instead of a complicated real-life situation.
This is a process of abstraction:
Concentrate on a few key properties.
Ignore all the other information.
The art of modeling is to make wise decisions about which properties to keep and which to discard.
In our billiard story:

## What are some of those simplifying assumptions?

Positions of cue ball and red ball are given. Then our first goal is to write instructions:

What direction to aim so that cue ball will hit object ball after a bounce against the top rail?

## REFLECTION RULE:

Angle of incidence ( $\alpha$ ) equals Angle of reflection ( $\alpha^{\prime}$ ).


We won't try explain why billiard balls bounce that way. For us, that "reflection rule" is a fact from physics.

Amazing observation: Reflection Rule also holds for rays of light hitting a flat mirror.
"Mirrors" prompt us to think about reflections across a line. Let's sketch our table (green) and its reflection (white) across the top edge.


The insight: REFLECTION RULE is equivalent to

## STRAIGHT-LINE RULE:

Paths from cue ball to rail to reflection of red ball lie on a straight line.


That new interpretation of the reflection rule answers our question:

WHERE TO AIM?

## Aim at the reflection of the red ball.

- Why are those rules equivalent? Here's a proof.

Green rectangle = original table White rectangle $=$ reflection of original table across its top rail $E F$.
$B^{\prime}=$ reflection of red ball $B$.
Choose point $R$ on segment $E F$, Then $R B^{\prime}=$ reflection of $R B$.

Symmetry of reflections implies the two angles marked $\alpha^{\prime}$ have equal measure.


ASSUME: Straight-line Rule: $A, R, B^{\prime}$ are on a line.

TO SHOW:

$$
\alpha=\alpha^{\prime}
$$

$A R B^{\prime}$ and $E R F$ are straight lines. "Vertical angles" are equal. Therefore $\alpha=\alpha^{\prime}$.

Done.


TO SHOW:
$A, R, B^{\prime}$ are on a line.

Let $\theta=$ measure of $\angle A R B$. Then Along line ERF:

$$
\alpha+\theta+\alpha^{\prime}=180^{\circ} .
$$

Then, $\theta+2 \alpha=180^{\circ}$
Then $\angle A R B^{\prime}=\theta+2 \alpha=180^{\circ}$.
Therefore
$A R B^{\prime}$ is a straight line.
Done.


Let's try a kick shot with the cue ball bouncing against the rail on the right.

Where to aim to make this happen?
Reflect the whole picture across that right edge.


Reflect the whole picture across that right edge.


Aim along the line from the cue ball to

## Two-bounce kick shots?

Where should we aim so that the cue ball will bounce against two rails and then hit the red ball?

Mirror method:
Draw four reflected copies of the original table.
Single reflections: one above and one to the right.
The fourth tile (upper right) is two reflection-steps from the original.



Aim at the second-reflection red ball in the upper right. The straight dotted line between those balls crosses two edge lines, indicating two bounces in the original table.

The three segments of that dotted straight line correspond to the three sections of the broken-line path within the original table.

This mirror method provides information about all possible shots sending cue ball to red ball.

STRATEGY: From the original picture of the table, create an "unfolded" reflection diagram.

Bounces on the original table correspond to grid lines the path crosses in the unfolded diagram.

## A counting question:

How many two-bounce shots send cue ball to red ball?

Sketch them.

In the next picture, the original table is the green-shaded rectangle in the center (with a white cue ball). Reflected images of that table will tile the whole plane with rectangles. Each tile contains one image of the red ball.

Our picture shows the original table and several reflected images.


Draw a line from the cue ball to one of the red balls. Reinterpret that straight segment as a broken-line path in the original table.
For the cue ball to hit the red ball:
How any one-bounce paths are there?
How any two-bounce paths are there?

In the next picture, some tiles are shaded.
Yellow = one step away from the center (one-bounce shots).
Blue = two steps away (two-bounce shots).


PROBLEMS for you to investigate:

1. Impossible shots?

With the cue ball on the table, where can we place the red ball so that a one-bounce shot against the top rail cannot be done?
2. We counted 4 one-bounce shots, and 8 two-bounce shots. Draw pictures of those shots on the original table. How many 3-bounce shots are there?

How many with $n$ bounces?
3. Shoot cue ball at a $45^{\circ}$ angle on a table of size $a \times b$.

Will the ball return to its starting point and retrace its path over and over?

On this $3 \times 1$ table, our path repeats after 8 bounces.

Why does an 8-bounce pattern happen for every starting position of the ball on that table?
(What if the ball hits a corner?)


For $45^{\circ}$ path on an $a \times b$ table.
Draw pictures on graph paper, for $45^{\circ}$ lines on various table sizes.
A $1 \times 1$ table has a 4 -bounce pattern.
What about a $1 \times 2$ table? A $2 \times 3$ table? Or $3 \times 5$ ?
If $a$ and $b$ are positive integers, explain why the path must repeat.

Bounce-Number: Sketch several different integer sizes $a \times b$.
Make a guess for the number of bounces per cycle.
Explain why your conjecture is correct.

Note: A $4 \times 6$ table has the same behavior as a $2 \times 3$ table. Explain why the bounce-number per cycle depends only on the ratio $a / b$.
4. If ratio $a / b$ is irrational, then:

A $45^{\circ}$ path bouncing on an $a \times b$ table never repeats.
For instance, this happens on a table of size $1 \times \sqrt{2}$.
Explain why that happens.
It's difficult to draw accurate bounce patterns to illustrate that non-repeating behavior. Any drawing (even with computer software) involves small errors.
The path will get very close to earlier sections of itself, but we know it can never repeat exactly.
5. How can we analyze paths at other angles?

Key idea: Draw a $45^{\circ}$ path in an $a \times b$ table. Stretch the picture horizontally to get paths at other angles.


Must the new picture satisfy the Reflection Rule?
Does every billiard path arise by stretching a $45^{\circ}$ path?

# THANKS FOR LISTENING. 

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