Geometry and Complex Numbers

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A transformation T of the plane is a **rigid motion** if it preserves distances.
For a point p we write $T(p)$ or Tp for the image of p after applying T.
"Preserving distances" means that, for any points a, b, then:

$$dist(Ta, Tb) = dist(a, b).$$

A figure $\mathcal{F}$ is transformed to another figure $\mathcal{F}' = T(\mathcal{F})$, and those figures $\mathcal{F}$ and $\mathcal{F}'$ look the same: distances and angles in $\mathcal{F}$ match those in $\mathcal{F}'$.

Examples.

A **translation** is rigid.
The transformation T :

"move two steps East and one step North"

is an example of a translation.

A **rotation** about a point is rigid.
For instance, rotate the plane 30° about the point O.

A **reflection** across line ℓ is rigid.
For instance, sending $(x, y) \mapsto (x, -y)$ is the reflection across the x-axis.

We usually use coordinates to represent points in the plane.
Choose an origin O and use two perpendicular lines through O to be the x and y axes. Each axis is a number line.

Ordered pair $p = (3, 2)$ represents the point found by:
start from O,
move 3 units East (direction of x-axis),
then move 2 units North (direction of y-axis). End at point p.

Distance traveled O to p is $\sqrt{3^2 + 2^2} = \sqrt{13}$.
We write $|p|$ for that distance:
Then $|p| = dist(O, p) = |(3, 2)| = \sqrt{13}$.

What's the distance from $p = (13, 5)$ to $q = (7, 2)$? It is:

$$dist(p, q) = |p - q| = |(6, 3)| = \sqrt{6^2 + 3^2} = \sqrt{45}.$$

Then by definition: transformation T is a rigid motion if:
For any points a, b we have: $|Ta - Tb| = |a - b|$.

The ideas of plane geometry and rigid motions go back to the ancient Greeks, as codified in Euclid's *Elements* in 300 BC.
Euclid defined two figures $\mathcal{F}$ and $\mathcal{F}'$ to be **congruent** if
there is some rigid motion T such that $\mathcal{F}' = T(\mathcal{F})$.

Euclid did not know any algebra.
That mathematical tool was developed by Arabian and Persian mathematicians over several centuries. Algebraic ideas were brought from Arabs to Europeans during the "renaissance" via Italy and Spain.

The **quadratic formula** was known since early times, but the "cubic formula" was discovered in Italy around 1450. Those formulas for solving equations led to the introduction of complex numbers.

Can "imaginary" quantities like $i = \sqrt{-1}$ have actual meaning?

Over the years, numbers like $2 + 3i$ and $12 - 7i$ were used more frequently. In the 1700s many mathematicians used complex numbers within calculus, expecially for working with infinite series.

That work led Euler to his famous identity: $e^{i\pi} = -1$.

Let's review basic operations for **complex numbers**.
Those have the form $a + bi$, where a, b are real numbers (that is, on the number line):

$(5 + 2i) + (3 + i) = 8 + 3i$.

$(5 + 2i) \cdot (3 + i) = 13 + 11i$.

(Because $(5 + 2i) \cdot (3 + i) = 5 \cdot 3 + 2 \cdot (-1) + 5 \cdot i + (2i) \cdot 3 = 13 + 11i$.)

Division? Express fraction $(5 + 2i)/(3 + i)$ in the form $a + bi$.

$$\frac{5 + 2i}{3 + i} = \frac{(5 + 2i)(3 - i)}{(3 + i)(3 - i)} = \frac{17 + i}{10} = \frac{17}{10} + \frac{1}{10}i.$$ This has the form $a + bi$.

We multiplied denominator $a + bi$ by its "conjugate" $a - bi$ to obtain a real number:

$$(a + bi)(a - bi) = a^2 + b^2.$$

Definition. If $z = x + yi$ define $\bar{z} = x - yi$.

Then $z\bar{z} = x^2 + y^2$ is real. (It has zero imaginary part).

Key property of complex conjugate:

Lemma. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

Algebraic proof: Suppose $z = a + bi$ and $w = c + di$.

$$z \cdot w = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

$$\overline{z} \cdot \overline{w} = (a - bi)(c - di) = (ac - bd) - (ad + bc)i.$$

□

Change of Viewpoint:
Around 1800 Gauss viewed the set of complex numbers geometrically.
Plot $z = x + yi$ as the point (x, y) in the **complex plane**.

Then the distance from O to z is: $|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$.

Corollary. $z\overline{z} = |z|^2$ and:
$|z \cdot w| = |z| \cdot |w|$. **Law of Moduli.**

$|z \cdot w| \stackrel{?}{=} |z| \cdot |w|$. Proof: $|z \cdot w|^2 = (zw)(\overline{zw}) = zw\bar{z}\,\overline{w} = (z\bar{z})(w\overline{w}) = |z|^2 \cdot |w|^2$.

When $z = i$ this says: $|iw| = |w|$.

Let S be the transformation: "multiply by i". That is $S(z) = iz$.

Claim. S is a rigid motion. (Because $|S(w)| = |w|$ for every w.)

Since $S(0) = 0$ we suspect: S is a rotation about O.
Since $S(1) = i$ we can see:

$$\boxed{S(z) = iz \text{ is the rotation about } O \text{ through } 90^\circ.}$$

In coordinates, for $z = x + yi$, then: $S(z) = iz = i(x + yi) = -y + xi$.

Given angle θ, let u be the point on unit circle with angle from 1 to u equal to θ. (It turns out that $u = \cos\theta + i\sin\theta$.)
Define transformation S_θ as "multiply by u": $S_\theta(z) = uz$.

Lemma. S_θ is rotation about O through angle θ.

An amazing connection between the algebra of complex numbers and the geometry of the plane.

How to represent other rotations using complex numbers?

Let T be the rotation about point p through angle θ.
Enable this in three steps:

Shift p to O, rotate about O, shift O back to p.

The shifts are: $z \mapsto z - p$ and $z \mapsto z + p$.
As before, let u be the unit circle point at angle θ. Then:

$$T(z) = u \cdot (z - p) + p.$$

Then that rotation is: $\boxed{T(z) = uz + p(1 - u).}$

Reverse this process:
Suppose $|u| = 1$ has angle θ. Then transformation

$$T(z) = uz + v$$

is the rotation through angle θ about point p,
where p satisfies $v = p(1 - u)$. Then $p = \frac{v}{1-u}$.

This all works well when $u \neq 1$. That is, when angle $\theta \neq 0$.

What if $u = 1$? Then $T(z) = z + v$ is a translation (NOT a rotation). Point $p = v/0 = \infty$.

Can we view a translation as a sort of rotation about ∞ . . . ? That's weird.

Here's a puzzle.
Suppose we first rotate the plane about O through 90°, and then rotate the plane about the point 6, also through 90°.

Is that composition of two actions another rotation?

What's its center point?

The first operation T is $T(z) = iz$.
The second operation S is $S(z) = iz + 6(1 - i)$.

Compose them, to obtain:

$$S(T(z)) = S(iz) = i(iz) + 6(1 - i) = -z + 6(1 - i).$$

This is a rotation. Its center p that is not moved: $S(T(p)) = p$.
Solve: $p = -p + 6(1 - i)$.
We find $2p = 6(1 - i)$ so that $p = 3(1 - i)$.

ANSWER: That composite is a 180° rotation about the point $3 - 3i$.

Must the composition of any two rotations be another rotation?

Related ideas to investigate.

Is every rigid motion of the plane equal to a
translation, rotation about a point, or reflection across a line ?

No. There are also "glide-reflections."

The "bar" transformation $B(z) = \bar{z}$ sends $x + yi \mapsto x - yi$.
This tranformation is: Reflect across the real axis.

⋆ Is every reflection across a line expressible as
$F(z) = u\bar{z} + v$ for some complex u, v with $|u| = 1$?.

⋆ If $|u| = 1$, does $F(z) = u\bar{z} + v$ represent a reflection across a line?

⋆ **Lemma.** Compose two reflections :: Get a rotation or translation.

Prove this with complex numbers. Explain it geometrically.

It's interesting to reflect on reflections.

⋆ Is there an algebraic way to represent rotations in 3-space?

In the 1840s, W.R. Hamilton answered by discovering "quaternions," extending the concept of complex numbers.

A fascinating algebraic way to study rotations in 3 (and 4) dimensions.

⋆ How did the study of quaternions, vectors, and rotation groups evolve and change over the past 180 years?

This wonderful topic leads to many advanced mathematical ideas.

⋆ Major ideas to explore:

How can we investigate the geometry of translations, rotations, reflections in n dimensions?

Interplays among
n dimensional geometry, algebra, and calculus in n variables.

Many questions remain unanswered . . .

This reminds me of a quote from Michael Atiyah, one of the greatest mathematicians of the 20th Century.

Algebra is an offer made by the devil. The devil says

"Here is a powerful machine that will answer many questions.
All you need to do is give me your soul:

Give up geometry and I will give you
this marvelous algebraic machine."

THANKS FOR LISTENING.

To ask mathematical questions, write to me at

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